## Appendix

## A Proofs for Section 2 (Preliminaries)

*Proof of proposition 2.* In the original paper presenting the concept of necessary winner [24], the authors use a different representation of partial profiles. Where we use partial rank matrices, they use partial orders on candidates.

This does not impact the characterisation of necessary winner which is independent of the representation. Indeed, for a profile $\mathcal{P}$, a voting rule $F$ and a winning candidate $c \in F(\mathcal{P})$, if for all candidate $c'$ different from $c$ and all $\mathcal{P}'$ a such that $\mathcal{P}$ is an extension of $\mathcal{P}'$, we have that $\sigma^{min}_{\mathcal{P}'}(c) \geq \sigma^{max}_{\mathcal{P}'}(c')$, then $\sigma_{\mathcal{P}'}(c) \geq \sigma_{\mathcal{P}'}(c')$ because by definition we have $\sigma_{\mathcal{P}'}(c) \geq \sigma^{min}_{\mathcal{P}'}(c)$ and $\sigma^{max}_{\mathcal{P}'}(c') \geq \sigma_{\mathcal{P}'}(c')$.

Konczak and Lang [24] present this result as an equivalence. However as pointed out by Xia and Conitzer [38] this is a mistake as the following counterexample shows. Consider four candidates $A$, $B$, $C$, $D$, three voters with the Borda rule. The first vote is the partial order $A \succ B$, $A \succ C$, $A \succ D$, $B \succ C$, $B \succ D$. The second vote is $A \succ C$, $A \succ D$, $B \succ A$, $B \succ C$, $B \succ D$. The third is $A \succ B$. $A$ is in first position in vote 1, in second in vote 2 and cannot be lower than third in vote 3. Hence, $\sigma^{min}(A) = 6$. Similarly, $\sigma^{max}(B) = 7$, $\sigma^{max}(C) = 5$ and $\sigma^{max}(D) = 5$. $A$ ties with $B$ in the first two votes but beats $B$ in the third vote so $A$ is ranked above $B$. Since $\sigma^{min}(A) > \sigma^{max}(C)$ and $\sigma^{min}(A) > \sigma^{max}(D)$, $A$ is ranked above $C$ and $D$. Thus, we know that $A$ will always be first and hence is a necessary winner. However, we have $\sigma^{min}(A) < \sigma^{max}(B)$.

Even if the equivalence does not hold in the original setting of partial orders, we now show that it holds when preference profiles are represented as rank matrices. Indeed, in our context, minimizing the score of a necessary winner $w$ and maximizing the score of any other candidate $c$ are two independent goals. Hence, given a partial rank matrix $\mathcal{R}$, for every candidate $c$, there exist a complete rank matrix $\mathcal{R}_c$ such that $\mathcal{R} \subseteq \mathcal{R}_c$, $\sigma_{\mathcal{R}_c}(w) = \sigma^{min}_{\mathcal{R}}(w)$ and $\sigma_{\mathcal{R}_c}(c) = \sigma^{max}_{\mathcal{R}}(c)$. Since $w$ is a necessary winner of partial rank matrix $\mathcal{R}$, for any extension $\mathcal{R}'$ of $\mathcal{R}$, we have $\sigma_{\mathcal{R}'}(w) \geq \sigma_{\mathcal{R}'}(c)$. Applying this to each $\mathcal{R}_c$, we have $\forall c \in \mathcal{C}$, $\sigma^{min}_{\mathcal{R}}(w) \geq \sigma^{max}_{\mathcal{R}}(c)$. To construct such extension of $\mathcal{R}$, we work ballot by ballot. If at least $w$ or $c$ is locked achieving both goals simultaneously then, is trivial. If both $w$ and $c$ are free, there are at least two free entries in the ballot. Hence, we can put $c$ in the most preferred free entry and $w$ in the least preferred free one without any problem. □

## B Proofs for Section 4 (Smallest abductive explanations for Borda)

**Theorem 4.** *Let $\mathcal{R}$ be a rank matrix with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$. For all AXp $\mathcal{X}$ of $\mathcal{R}$, we have:*

$$|\mathcal{X}| \geq n - \left\lfloor \frac{n}{m} \right\rfloor \tag{6}$$

Building on notations introduced in Section 2.4, for a given partial rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$, we define the score margin of $c$ w.r.t. $c'$ for $\mathcal{R}_i$ as $\delta^c_{\mathcal{R}_i}(c') = \sigma^{min}_{\mathcal{R}_i}(c) - \sigma^{max}_{\mathcal{R}_i}(c')$ and the margin of victory of $c$ w.r.t. $c'$ for $R$ as $\delta^c_{\mathcal{R}}(c') = \sum_{\mathcal{R}_i \in \mathcal{R}} \delta^c_{\mathcal{R}_i}(c')$. The inequality of Proposition 2 can now be rewritten as $\delta^c_{\mathcal{R}}(c') \geq 0$.

Additionally, we introduce the total score margin of $c$ as the sum of the score margin of $c$ w.r.t. all other candidates, $\Delta^c_{\mathcal{R}_i} = \sum_{c' \neq c} \delta^c_{\mathcal{R}_i}(c')$. Similarly, we have the total margin of victory of $c$, $\Delta^c_{\mathcal{R}} = \sum_{c' \neq c} \delta^c_{\mathcal{R}}(c')$.

Note that if $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$ then $\Delta^w_{\mathcal{R}} \geq 0$ since for all other candidates, $c$, $\delta^w_{\mathcal{R}}(c) \geq 0$.

Throughout the rest of this section, we will consider the margins of the winning candidate $w$. Hence, the dependency on $w$ of $\delta^w$ and $\Delta^w$ will be dropped.

We first introduce two useful lemmas.

**Lemma 6.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$.*

- *If $w \in \mathcal{R}_i$ and $\mathcal{R}_{i,1} = null$ then*
$$\mathcal{R}'_{i,k} = \begin{cases} w & \text{if } k = 1 \\ \mathcal{R}_{i,k} & \text{if } k \neq k_w \text{ with } k_w \text{ s.t. } \mathcal{R}_{i,k_w} = w \end{cases}$$
*satisfies $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$ and $|\mathcal{R}_i| = |\mathcal{R}'_i|$.*
- *If $w \in \mathcal{R}_i$ and $\mathcal{R}_{i,1} \neq null$ then*
$$\mathcal{R}'_{i,k} = \begin{cases} w & \text{if } k = 1 \\ \mathcal{R}_{i,1} & \text{if } k = k_w \text{ with } k_w \text{ s.t. } \mathcal{R}_{i,k_w} = w \\ \mathcal{R}_{i,k} & \text{if } k \neq 1 \text{ and } k \neq k_w \end{cases}$$
*satisfies $\Delta_{\mathcal{R}'_i} \geq \Delta_{\mathcal{R}_i}$ and $|\mathcal{R}_i| = |\mathcal{R}'_i|$.*
- *If $w \notin \mathcal{R}_i$ and $\mathcal{R}_{i,1} \neq null$ then*
$$\mathcal{R}'_{i,k} = \begin{cases} w & \text{if } k = 1 \\ \mathcal{R}_{i,k} & \text{if } k \neq 1 \end{cases}$$
*satisfies $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$ and $|\mathcal{R}_i| = |\mathcal{R}'_i|$.*

*Proof.* If for all $c \in \mathcal{C} \setminus \{w\}$, $\delta_{\mathcal{R}'_i}(c) \geq \delta_{\mathcal{R}_i}(c)$ and there exists $c_0 \in \mathcal{C} \setminus \{w\}$, $\delta_{\mathcal{R}'_i}(c_0) > \delta_{\mathcal{R}_i}(c_0)$ then $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$.

In the first case, for all $c \in dom(\mathcal{R}_i) \setminus \{w\}$, $\sigma^{max}_{\mathcal{R}'_i}(c) = \sigma^{max}_{\mathcal{R}_i}(c)$ and for all $c \notin dom(\mathcal{R}_i) \setminus \{w\}$, $\sigma^{max}_{\mathcal{R}'_i}(c) < \sigma^{max}_{\mathcal{R}_i}(c)$ (the first entry, once free in $\mathcal{R}_i$ is not anymore in $\mathcal{R}'_i$). Additionally, $\sigma^{min}_{\mathcal{R}'_i}(w) > \sigma^{min}_{\mathcal{R}_i}(w)$. Hence, for all $c \in \mathcal{C} \setminus \{w\}$, $\delta_{\mathcal{R}'_i}(c) > \delta_{\mathcal{R}_i}(c)$ and $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$. The second and third cases are analogous. □

We proved that in the optimal case for total margin of victory, if the first entry is locked it will only be with the winner and if the winner is locked it will only be in first entry. Now, let us turn our attention to the rest of the ranking.

**Lemma 7.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$. If there exists $c \in \mathcal{R}_i \setminus \{w\}$ and $k_0, k_1 \in [|1, m|]$ s.t. $\mathcal{R}_{i,k_0} = null$, $\mathcal{R}_{i,k_1} = null$ and $k_0 < k_c < k_1$ with $k_c$ s.t. $\mathcal{R}_{i,k_c} = c$ then*

$$\mathcal{R}'_{i,k} = \begin{cases} c & \text{if } k = k_1 \\ \mathcal{R}_{i,k} & \text{if } k \neq k_c \end{cases}$$

*satisfies $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$ and $|\mathcal{R}_i| = |\mathcal{R}'_i|$.*

*Proof.* Similarly to the previous proof, we compare for all $c' \in \mathcal{C} \setminus \{w\}$, $\delta_{\mathcal{R}'_i}(c')$ and $\delta_{\mathcal{R}_i}(c')$. If $w \in ran(\mathcal{R}_i)$, for all $c' \in \mathcal{C} \setminus \{w, c\}$, $\delta_{\mathcal{R}'_i}(c') = \delta_{\mathcal{R}_i}(c')$ and $\delta_{\mathcal{R}'_i}(c) > \delta_{\mathcal{R}_i}(c)$. Otherwise, for all $c' \in \mathcal{C} \setminus \{w, c\}$, $\delta_{\mathcal{R}'_i}(c') > \delta_{\mathcal{R}_i}(c')$ and $\delta_{\mathcal{R}'_i}(c) > \delta_{\mathcal{R}_i}(c)$ ($\sigma^{min}_{\mathcal{R}'_i}(w) > \sigma^{min}_{\mathcal{R}_i}(w)$). Finally, for all $c \in \mathcal{C} \setminus \{w\}$, $\delta_{\mathcal{R}'_i}(c) > \delta_{\mathcal{R}_i}(c)$. Hence, $\Delta_{\mathcal{R}'_i} > \Delta_{\mathcal{R}_i}$. □

Note that the previous lemma does not hold if there is no free entry before the locked entry . Indeed, otherwise, lowering the candidate entry will improve the first free entry and free candidates could end up with a better score margin. Hence, the presence of $k_0$.

We are ready to show that a specific pattern of locked entries dominate any other pattern of the same size.

**Definition 7.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$. If there exists $k_1, k_2 \in [|1, m|]$ such that $\mathcal{R}_{i,k} = null$ if and only if $k_1 < k < m + 1 - k_2$, we say that $\mathcal{R}_i$ is in* ***normal form*** *of parameter $k_1$ and $k_2$. When $k_1 + k_2 = m$, we take $k_1 = m$ and $k_2 = 0$.*

**Lemma 8.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$. There exists $\mathcal{R}'_i$ in normal form of parameter $k_1$ and $k_2$ with $k_1 + k_2 = |\mathcal{R}_i|$ s.t. $\Delta_{\mathcal{R}'_i} \geq \Delta_{\mathcal{R}_i}$.*

*Proof.* For every ballot $\mathcal{R}_i$, repeated applications of the transformations given in lemmas 6 and 7 lead to the profile $\mathcal{R}'_i$ with the given normal form. ☐

Now that we have introduced the normal form, it is possible to evaluate its total margin of victory.

**Lemma 9.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$. Assuming there exists $k_1, k_2 \in \mathbb{N}$ s.t. $\mathcal{R}_i$ is in normal form of parameter $k_1$ and $k_2$.*

- *If $k_1 > 0$, $\Delta_{\mathcal{R}_i} = (k_1 + k_2)\left(m - \frac{(k_1+k_2+1)}{2}\right)$.*
- *If $k_1 = 0$, $\Delta_{\mathcal{R}_i} = -(m - 1 - k_2)^2 + \frac{k_2(k_2+1)}{2}$.*

*Proof.* In the first case, $\Delta_{\mathcal{R}_i} = \sum_{c \neq w} \delta_{\mathcal{R}_i}(c) = \sum_{i=1}^{k_1-1} i + (m - k_1 - k_2)k_1 + \sum_{i=m-k_2}^{m-1} i$ and in the second, $\Delta_{\mathcal{R}_i} = 0 - (m - 1 - k_2)(m - 1 - k_2) + \sum_{i=1}^{k_2} i$ where the first term is accounting for the candidates locked at the top of the ranking, the second, the free candidates and the third, the candidates locked at the bottom of the ranking. ☐

Finally, we can give an upper bound on the total margin of victory of any ballot.

**Lemma 10.** *Let $\mathcal{R}_i$ be a ballot of a rank matrix $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$ and $|\mathcal{R}_i| \geq 1$*

$$\Delta_{\mathcal{R}_i} \leq (m-1)(m|\mathcal{R}_i| - (m-1))$$

*Proof.* Thanks to Theorem 8, we know that ballots in normal form dominate other patterns in term of total margin of victory for a given number of locked entries. Thus, we just have to show that normal form ballots verify the inequality.

We derive the result from an upper bound on the score contribution per locked entry, $\frac{\Delta_{\mathcal{R}_i} - \Delta_\varnothing}{|\mathcal{R}_i|} \leq m(m-1)$ with $\Delta_\varnothing = -(m-1)^2$ the total margin of victory of the empty partial ballot.

Let $\mathcal{R}_i$ be a ballot in normal form of parameter $k_1$ and $k_2$ s.t. $k_1 + k_2 \geq 1$.

- If $k_1 > 0$, $\frac{\Delta_{\mathcal{R}_i} - \Delta_\varnothing}{|\mathcal{R}_i|} = \frac{(m-1)^2}{k_1+k_2} + m - \frac{(k_1+k_2+1)}{2}$.
  Clearly, this is less than $m(m-1)$ with equality for $k_1 = 1$ and $k_2 = 0$.
- If $k_1 = 0$, $\frac{\Delta_{\mathcal{R}_i} - \Delta_\varnothing}{|\mathcal{R}_i|} = 2m - \frac{k_2}{2} - \frac{3}{2}$.
  Since $k_2 \geq 1$, we have the desired result. ☐

*Proof of theorem 4.* Let $\mathcal{R}$ be a rank matrix with $n$ voters and $m$ candidates s.t. $w \in \mathcal{C}$ is a Borda winner of $\mathcal{R}$.

Assuming there exists $\mathcal{X}$, an AXp of size at most $n - \left\lfloor \frac{n}{m} \right\rfloor - 1$. Since $w \in \mathrm{NW}_{\mathrm{Borda}}(\mathcal{X})$, we have $\Delta_\mathcal{X} \geq 0$.

However, we have:

$$\begin{aligned}
\Delta_\mathcal{X} &= \sum_{\mathcal{X}_i \in \mathcal{X}} \Delta_{\mathcal{X}_i} \\
&\leq \sum_{\mathcal{X}_i \in \mathcal{X}} (m-1)(m|\mathcal{R}_i| - (m-1)) \\
&= (m-1)\left(m \sum_{\mathcal{X}_i \in \mathcal{X}} |\mathcal{R}_i| - \sum_{\mathcal{X}_i \in \mathcal{X}} (m-1)\right) \\
&\leq (m-1)\left(m\left(n - \left\lfloor \frac{n}{m} \right\rfloor - 1\right) - n(m-1)\right) \\
&= (m-1)\left(n - \left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right) m\right) \\
&< 0
\end{aligned}$$

Contradiction. Hence, no AXp of size at most $n - \left\lfloor \frac{n}{m} \right\rfloor - 1$ exists for any rank matrix. ☐

**Theorem 5.** *Let $\mathcal{R}$ be a rank matrix with $n$ voters and $m$ candidates and $w \in \mathcal{C}$ a Borda winner of $\mathcal{R}$ s.t. $\mathcal{X} \subseteq \mathcal{R}$ where $\mathcal{X}$ is the partial rank matrix where $w$ is ranked first for $n - \left\lfloor \frac{n}{m} \right\rfloor$ ballots and every other entry is null. $\mathcal{X}$ is an AXp of $\mathcal{R}$ and $|\mathcal{X}| = n - \left\lfloor \frac{n}{m} \right\rfloor$.*

*Proof.* Let $\mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be the partial rank matrix where every entry is $null$ except for $i \in [|1, n - \left\lfloor \frac{n}{m} \right\rfloor|]$ where $\mathcal{X}_{i,1} = w$. For all $c \in \mathcal{C} \setminus \{w\}$, we have:

$$\begin{aligned}
\delta_\mathcal{X}(c) &= \sum_{\mathcal{X}_i \in \mathcal{X}} \delta_{\mathcal{X}_i}(c) \\
&= \sum_{i=1}^{n - \left\lfloor \frac{n}{m} \right\rfloor} \delta_{\mathcal{X}_i}(c) + \sum_{i=n-\left\lfloor \frac{n}{m} \right\rfloor + 1}^{n} \delta_{\mathcal{X}'_i}(c) \\
&= \left(n - \left\lfloor \frac{n}{m} \right\rfloor\right) 1 - \left\lfloor \frac{n}{m} \right\rfloor (m-1) \\
&= n - \left\lfloor \frac{n}{m} \right\rfloor m \\
&\geq 0
\end{aligned}$$

Hence, $w \in \mathrm{NW}_{\mathrm{Borda}}(\mathcal{X})$. Since there is no smaller AXp (Theroem 4), $\mathcal{X}$ is an AXp of $\mathcal{R}$ and $n - \left\lfloor \frac{n}{m} \right\rfloor$ locked entries suffice. ☐

# C Data set composition of experiment on SiAXps

| model | parameter | #profiles |
|---|---|---|
| Impartial Culture | | 10 |
| Impartial Anonymous Culture | | 10 |
| normalized Mallows | $\phi = 0.5$ | 10 |
| Urn model | $\alpha = 0.1$ | 10 |
| Single-Peaked (Conitzer) | | 10 |
| Single-Seaked (Walsh) | | 10 |
| Single-Peaked On a Circle | | 10 |
| Single-Crossing | | 10 |
| Group-Separable | | 10 |
| 1-Cube (Interval) | uniform interval | 10 |
| 3-Cube (Cube) | uniform cube | 10 |
| Disc | disc in 2D | 10 |
| Circle | circle in 2D | 10 |
| Sphere | sphere in 3D | 10 |
| Uniformity (UN) | | 4 |
| Identity (ID) | | 1 |
| Antagonism (AN) | | 1 |