Indirect shooting method

I) Statement of the optimal control problem and necessary conditions of optimality.

<u>Assumptions</u> (Notations) : Let $\dot{x}(t) = f(x(t), u(t))$ an autonomous control system with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ smooth (i.e at least $\mathcal{C}^1$).
Let $U \subset \mathbb{R}^m$ the control set (arbitrary).
Let consider $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ smooth, $t_f \geq 0$ (final time) and $x_0 \in \mathbb{R}^n$.
Let $c : \mathbb{R}^n \to \mathbb{R}^k$, $k \leq n$, be a smooth submersion on $c^{-1}(\{0\})$, (initial condition)
that is $c'(x)$ of full rank if $c(x) = 0$.

<u>We define the ocp:</u>

$$(P_{Lc}) \begin{cases} \min\limits_{u \in \mathcal{U}} J(u) = \displaystyle\int_0^{t_f} L(x(t), u(t))\, dt & \text{cost in (Lagrange form)} \\ \dot{x}(t) = f(x(t), u(t)),\ u(t) \in U,\ t \in [0, t_f] \text{ a.e},\ x(0) = x_0 \\ c(x(t_f)) = 0 \end{cases}$$

and we recall that:

* $\mathcal{U} = L^\infty([0, t_f], \mathbb{R}^m)$ Remark: $\underset{\text{continuous}}{\mathcal{C}^0} \subset \underset{\text{discontinuous on a finite set}}{\mathcal{C}^0_m} \subset \underset{\text{discontinuous on a null set}}{L^\infty}$ on $[0, t_f]$
* Fixing $u \in L^\infty$ and $x_0 \in \mathbb{R}^n$, $\exists!$ maximal solution of $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x_0$.
* Eq: $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x_0$ $\equiv$ Eq: $x(t) = x_0 + \displaystyle\int_0^{t_f} f(x(t), u(t))\, dt$

in the sense of Carathéodory. That is the sol is absolutely continuous and $\dot{x} = f$ a.e

<u>Remark</u>: the functional J depends only on u and not (x, u) since fixing u (with x_0) gives x, which will be mentionned as the <u>associated trajectory</u>.

<u>Necessary conditions of optimality.</u>

<u>Def</u>: A <u>control</u> $u \in \mathcal{U}$ is called admissible if $u(t) \in U$ $\forall t \in [0, t_f]$ and if the associated trajectory x satisfies $c(x(t_f)) = 0$.

It is called <u>optimal</u> if for any admissible control w we have $J(u) \leq J(w)$

The Pontryagin Maximum Principle states that if u is optimal, with x its associated trajectory, then there exists a covector $p : [0, t_f] \to \mathbb{R}^n$, a scalar $p^0 \in \{-1, 0\}$, a Lagrange multiplier $\lambda \in \mathbb{R}^k$ such that:

a) $(p, p^0) \neq (0, 0)$

b) $\dot{x}(t) = \nabla_p H(x(t), p(t), u(t))$, $\dot{p}(t) = -\nabla_x H(x(t), p(t), u(t))$ a.e on $[0, t_f]$

c) $H(x(t), p(t), u(t)) = \max\limits_{w \in U} H(x(t), p(t), w)$

with $H(x, p, u) = (p \mid f(x, u)) + p^0 L(x, u)$ (the pseudo-Hamiltonian)

d) $p(t_f) = J_c^T(x(t_f))\, \lambda = \displaystyle\sum_{i=1}^{k} \lambda_i \nabla c_i(x(t_f))$ (transversality condition)

Remark: if the target is fixed to $x_f \in \mathbb{R}^n$, that is if $c(x) = x - x_f$, then the transversality condition becomes $p(t_f) = \lambda$ since $J_c(x) = I$, and so this condition is useless and usually we forget λ.

If $U = \mathbb{R}^m$, we have the weaker necessary conditions: $\begin{cases} \frac{\partial H}{\partial u}(\dots) = 0 \\ \frac{\partial^2 H}{\partial u^2}(\dots) \leq 0 \end{cases}$ ← negative semi-definite

Assumptions:
* $U = \mathbb{R}^m$
* $\forall (x,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \mapsto H(x,p,u)$ has a unique maximum denoted $\varphi(x,p)$ or $u[x,p]$.
* φ is smooth, that is at least C^1.

Under these assumptions, $\nabla_u H(x,p,\varphi(x,p)) = 0 \quad \forall (x,p)$.

Example: $H(x,p,u) = H_0(x,p) + \sum_{i=1}^m u_i H_i(x,p) - \frac{1}{2}\|u\|_2^2 \Rightarrow u[x,p] = \Phi(x,p)$, with $\Phi = (H_1, \dots, H_m)$.

Counter-example: $H = H_0 + u H_1$, $|u| \leq 1 \Rightarrow u[x,p] = \begin{cases} 1 & \text{if } H_1(x,p) > 0 \\ -1 & \text{if } \quad " \quad < 0 \\ ? & \text{if } \quad " \quad = 0 \end{cases}$

The true Hamiltonian:

Let define $h(z) = H(z, \varphi(z))$, $z = (x,p)$, the maximized Hamiltonian, which is a true Hamiltonian.

Then, we have $h'(z) = \frac{\partial H}{\partial z}(z,\varphi(z)) + \underbrace{\frac{\partial H}{\partial u}(z,\varphi(z))}_{=0} \varphi'(z) = \frac{\partial H}{\partial z}(z,\varphi(z))$,

that is eq b) and c) are equivalent to $\dot{z}(t) = \vec{h}(z(t))$! , $\vec{h} = (\nabla_p h, -\nabla_x h)$

Extremals and BC-extremals

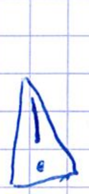
Def: (x, p, p^0, u) is called an extremal if it is solution of a, b and c.
It is called a BC-extremal if it is an extremal satisfying $x(0) = x_0$, $c(x(t_f)) = 0$ and d.

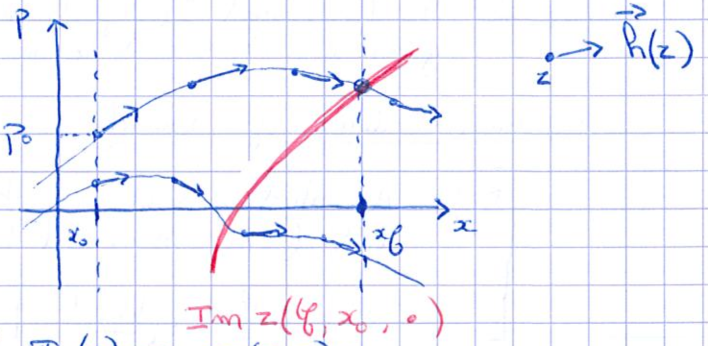
Illustration: $c(x) = x - x_f$:

goal: find p_0 such that

$\pi_x(z(t_f, x_0, p_0)) = x_f$,

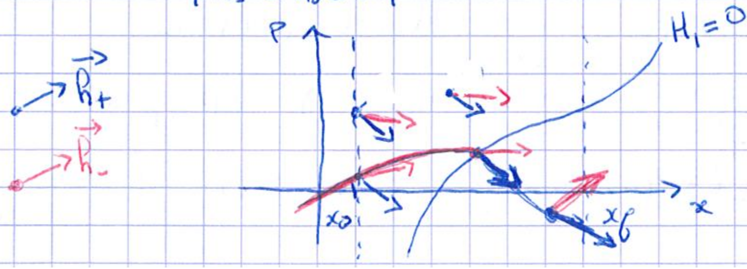
where $z(t_f, x_0, p_0)$ is the sol of $\dot{z} = \vec{h}(z)$, $z(0) = (x_0, p_0)$ and $\pi_x(z) = x$, $z = (x,p)$.





Counter-example: $H = H_0 + u H_1$, $|u| \leq 1$. Let (z,u) be an extremal such that $\{t \mid H_1(z(t)) = 0\}$ is finite $\Rightarrow u$ is bang-bang and switch between +1 and -1. $c(x) = x - x_f$.

We define $\begin{cases} h_+ = H_0 + H_1 \\ h_- = H_0 - H_1 \end{cases}$ the two true Hamiltonians in competition. We have:



II] Examples and boundary value problems.

1. Simple 1D example.

$$\begin{cases} \min J(u) = \frac{1}{2}\int_0^{1=t_f} u(t)^2\,dt, & \dot{x}(t) = -x(t) + u(t),\ u(t)\in\mathbb{R},\ t\in[0,1]\ p.p.\ a.e., \\ & x(0)=-1,\ x(1)=0. \end{cases}$$

$$H(x,p,u) = p(-x+u) + \frac{1}{2}p^0 u^2.$$

Necessary conditions: $\dot{x} = -x + u$

$\dot{p} = p$

$\frac{\partial H}{\partial u} = p + p^0 u = 0 \Rightarrow p^0 = -1$ else $(p,p^0) = (0,0)$.

$\Rightarrow$ we have to solve the Boundary Value Problem (BVP):

$$\begin{cases} \dot{x} = -x + p \\ \dot{p} = p \\ x(0) = -1 = x_0,\ x(1) = 0 = x_f. \end{cases} \quad \left\{ \dot{z}(t) = \vec{h}(z(t)),\ h(z) = -px + \frac{1}{2}p^2 \right.$$

$\Rightarrow p(t) = e^t p_0 \Rightarrow x(t) = p_0\,sh(t) + x_0 e^{-t}$

$\Rightarrow x(t_f) = x_f \iff p_0^* = \frac{x_f - x_0 e^{-t_f}}{sh(t_f)} = \frac{e^{-1}}{sh(1)} = \frac{2}{e^2-1} \simeq 0.3130.$

Remark: with p_0^*, we have $(x(t),p(t))$ by integrating $\dot{z} = \vec{h}(z)$, $z(0) = (x_0, p_0^*)$ and so we have $u(t) = p(t)$!

Remark: to compute p_0^*, we have solved the linear shooting equation:

⚠ $$S(p_0) \overset{def}{=} \pi(z(t_f, x_0, p_0)) - x_f = p_0\,sh(t_f) + x_0 e^{-t_f} - x_f,$$

where $z(\cdot, x_0, p_0)$ is the solution of $\dot{z} = \vec{h}(z)$, $z(0) = (x_0, p_0)$.

2. Calculus of variations.

$$\begin{cases} \min J(u) = \int_0^{t_f} L(x(t), u(t))\,dt, \\ \dot{x}(t) = u(t) \in \mathbb{R}^n, \quad t\in[0,t_f]\ a.e.,\ x(0) = x_0 \\ x(t_f) = x_f. \end{cases}$$

$$H(x,p,u) = (p\,|\,u) - L(x,u) \qquad (p^0 = -1)$$

Necessary conditions: $\dot{x} = u$

$\dot{p} = \nabla_x L(x,u)$

$0 = \nabla_u H = p - \nabla_u L(x,u)$

Remark: $p = \nabla_u L(x,u) \Rightarrow \dot{p} = \frac{d}{dt}\nabla_u L(x,u) = \nabla_x L(x,u)$: Euler-Lagrange equation.

We have to solve the BVP:

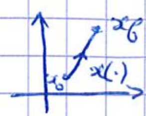
$$\begin{cases} \dot{x} = u \\ \dot{p} = \nabla_x L(x,u) \\ 0 = p - \nabla_u L(x,u) \\ x(0) = x_0,\ x(t_f) = x_f. \end{cases}$$

with the first three equations: DAE (Differential-Algebraic Equation)

<u>Particular cases:</u>

a) $L(x,u) = \frac{1}{2}(A(x)u\,|\,u) + (b(x)\,|\,u) + c(x)$ (with smooth data) with $A(x) > 0\ \forall x$. Then, $Q(u) = (p\,|\,u) - L(x,u) = H(x,p,u)$ has a unique maximum (strictly concave quadratic) given by $Q'(u) = 0$, that is $u[x,p] = A(x)^{-1}(p - b(x))$.

<u>Remark</u>: if $L(x,u) = \frac{1}{2}\|u\|^2$ then $u[x,p] = p$: this is the pb of computing the Euclidean distance between two points:



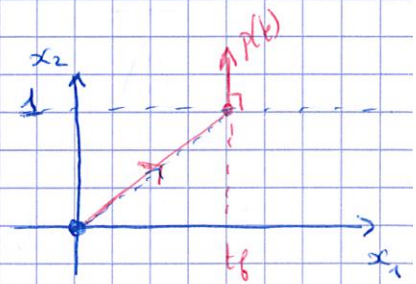
b) L smooth st $\forall x$, $u \mapsto L(x,u)$ strictly convex and s.t.

$$\lim_{\|u\| \to +\infty} \frac{L(x,u)}{\|u\|} = +\infty.$$

<u>Remark</u>. coercivity, that is $\lim_{\|u\|\to+\infty} L(x,u) = +\infty$ is not sufficient, take $L(x,u) = |u| \Rightarrow H(x,p,u) = \underbrace{pu - |u|}_{M(u)}$ has no maximum for $p = 2$.

3. "Navigation" problem.

$$\begin{cases} \min J(u) = \frac{1}{2}\int_0^{t_f} \|u\|^2\,dt \\ \dot{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u, \quad u \in \mathbb{R}^2, \quad x(0) = (0,0) \\ x_2(t_f) = 1. \end{cases}$$



$$H(x,p,u) = p_1 + (p\,|\,u) - \frac{1}{2}\|u\|^2 \qquad (p^0 = -1)$$

Necessary conditions: $\dot{x} = (1,0) + u$, $\dot{p} = 0$, $u = p$ $\quad + \quad p(t_f) = \lambda \nabla c(x(t_f)) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$, $c(x) = x_2 - 1$

$$\Rightarrow \text{BVP} \quad \begin{cases} \dot{x} = (1,0) + p \\ \dot{p} = 0 \\ x(0) = (0,0), \quad x_2(t_f) = 1, \quad p(t_f) = (0, \lambda) \end{cases}$$

$$\Rightarrow p(t) = (\alpha, \beta) \Rightarrow x(t) = \begin{pmatrix} \alpha+1 \\ \beta \end{pmatrix} t + \underbrace{x(0)}_{(0,0)}$$

$x_2(t_f) = 1 \iff \beta t_f = 1 \iff \beta = \frac{1}{t_f}$

$p(t_f) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \iff \lambda = \beta = \frac{1}{t_f}$ and $\alpha = 0$

$$\begin{cases} u(t) = p(t) = \begin{pmatrix} 0 \\ \frac{1}{t_f} \end{pmatrix}, \quad \lambda = \frac{1}{t_f} \\ x(t) = \begin{pmatrix} 1 \\ \frac{1}{t_f} \end{pmatrix} t \end{cases}$$

III] Indirect simple shooting

Under our assumptions, thanks to the PMP, we have to solve:

$$
\text{(BVP)}_{\text{-DAE}} \begin{cases} \dot{x}(t) = \nabla_p H(x(t), p(t), u(t)) \\ \dot{p}(t) = -\nabla_x H[t] \\ 0 = \nabla_u H[t] \\ x(0) = x_0, \quad c(x(t_f)) = 0, \quad p(t_f) = J_c(x(t_f))^T \lambda . \end{cases} \qquad [t] = (x(t), p(t), u(t))
$$

Since we can get the control in a smooth (at least $\mathcal{C}^1$) feedback form:

$$u(t) = u[x(t), p(t)]$$

then $\text{(BVP)}_{\text{-DAE}}$ becomes:

true Hamiltonian

$$
\text{(BVP)} \begin{cases} \dot{z}(t) = \vec{H}(z(t), u[z(t)]) = \vec{h}(z(t)) \\ x(0) = x_0, \quad c(x(t_f)) = 0, \quad p(t_f) = J_c(x(t_f))^T \lambda . \end{cases}
$$

We can write (BVP) as a system of nonlinear equations:

shooting function

$$
\begin{aligned} S : \mathbb{R}^n \times \mathbb{R}^k &\longrightarrow \mathbb{R}^k \times \mathbb{R}^n \\ (p_0, \lambda) &\longmapsto S(p_0, \lambda) = \begin{pmatrix} c \circ \pi_x(z(t_f, x_0, p_0)) \\ \pi_p(z(t_f, x_0, p_0)) - J_c(\pi_x(z(t_f, x_0, p_0)))^T \lambda \end{pmatrix} \end{aligned}
$$

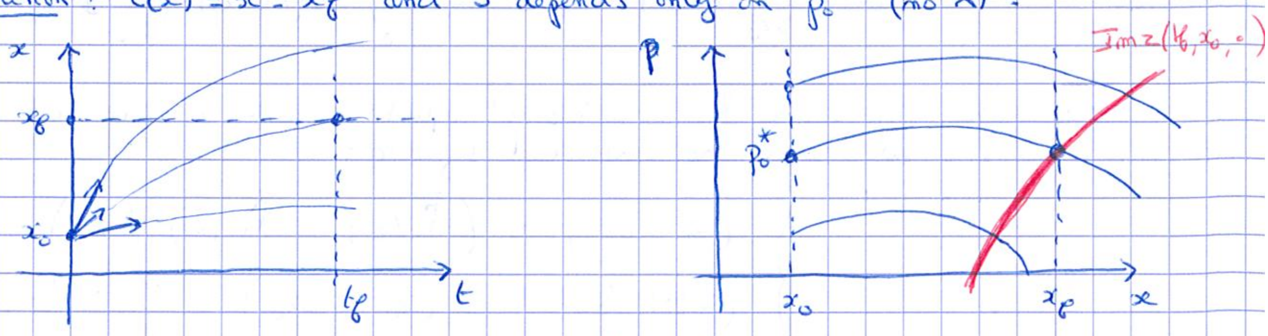
where $z(t, x_0, p_0)$ is the solution at time t, of $\dot{z} = \vec{h}(z)$, $z(0) = (x_0, p_0)$,

Hence, we need: $\qquad \pi_x(x,p) = x$ and $\pi_p(x,p) = p$.

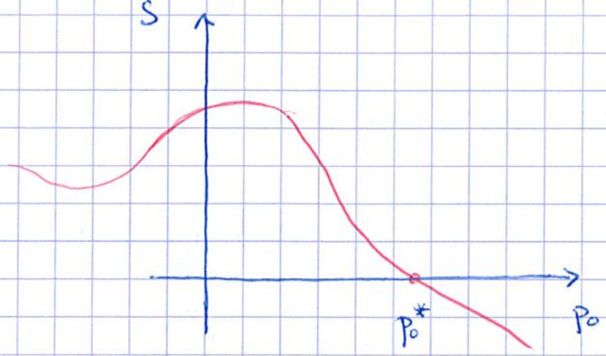
- A method to compute $z(t_f, x_0, p_0)$ by integration. We usually use a Runge-Kutta integrator with variable step-size.
- A Newton solver to solve $S = 0$. We recall that Newton solver are sensitive to the initial guess. A difficulty is thus to give an initial (p_0, λ) which makes the method converges.

Remark: Solving $S = 0$ is what we call the $\boxed{\text{indirect simple shooting method}}$. We can solve (BVP-DAE) or (BVP) by collocation method, discretizing the extremal (z, u) on a grid of times $t_0, t_1, \dots, t_N = t_f$. The grid may be refined iteratively.

Illustration: $c(x) = x - x_f$ and S depends only on p_0 (no λ).

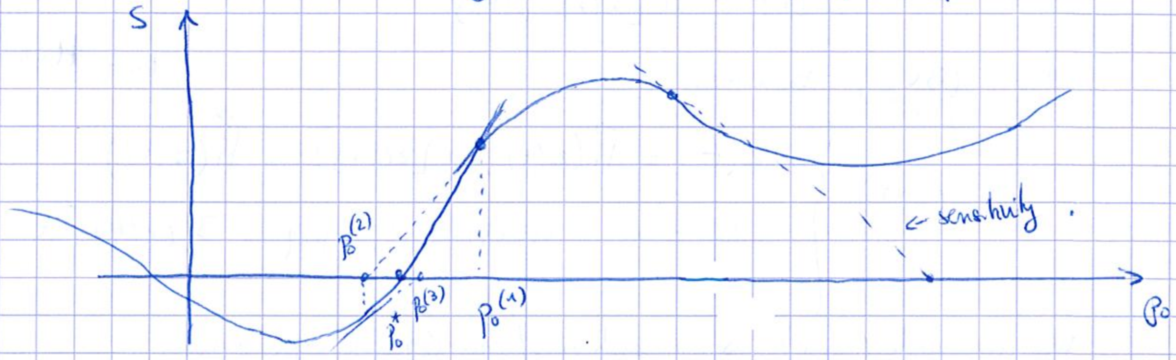


Goal: reach the target finding the right initial impulse p_0 = shooting.



$$S(p_0) = \pi_x\left(z(t_f, x_0, p_0)\right) - x_f .$$

Recalls on the Newton method: to solve the system of nonlinear equations, the Newton solver, solves iteratively a linearized version of the pb.



<u>Iteration of the Newton solver</u>:

$$p_0^{(k+1)} = p_0^{(k)} + d \quad , \text{ where } d \text{ is sol of } S'(p_0^{(k)}).d = -S(p_0^{(k)}) .$$

So we need the Jacobian of the shooting function

<u>Ex</u>: $c(x) = x - x_f$ (no λ) , $S(p_0) = \pi_x(z(t_f, x_0, p_0)) - x_f$

$$S'(p_0).d = \pi_x\left(\frac{\partial z}{\partial p_0}(t_f, x_0, p_0).d\right)$$
$$= \pi_x\left(\frac{\partial z}{\partial z_0}(t_f, x_0, p_0).(\underset{\mathbb{R}^n}{0}, d)\right) \quad , \quad z_0 = (x_0, p_0)$$

so we have to compute:

$$\frac{\partial z}{\partial z_0}(t_f, x_0, p_0).\delta z_0 \quad , \quad \delta z_0 = (0_{\mathbb{R}^n}, d) ,$$

solution of the variational equation:

$$\dot{\delta z}(t) = \vec{h}'(z(t, x_0, p_0)).\delta z(t) , \quad \delta z(0) = \delta z_0$$

<u>Remark</u>: If we have: $z(t, x_0, p_0) = z(t, z_0) = z_0 + \int_0^t \vec{h}(z(s, z_0))\, ds$

then $\frac{\partial z}{\partial z_0}(t, z_0).\delta z_0 = \delta z_0 + \int_0^t \vec{h}'(z(s, z_0)).\left(\frac{\partial z}{\partial z_0}(s, z_0).\delta z_0\right) ds$

<u>Remark</u>: To solve the variational equations with a numerical integrator, we solve the extended system:

$$\begin{cases} (\dot z, \dot{\delta z}) = (\vec{h}(z), \vec{h}'(z).\delta z) \\ (z(0), \delta z(0)) = (z_0, \delta z_0) . \end{cases}$$

this can be computed by Automatic Differentiation or finite differences.

Example of the computation of δz by eq var:

We consider $\vec{h}_1(z) = (-x+p\,,\,p)$, cf II.1.

$\Rightarrow$ $S(p_0) = p_0\,\mathrm{sh}(t_f) + x_0 e^{-t_f} - x_f$ $\Rightarrow$ $S'(p_0) = \mathrm{sh}(t_f)$.

Let us retrieve this result with the eq var:

$$\dot{\delta z} = \vec{h}_1{}'(z).\delta z = \underbrace{\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}}_{=A}\delta z = A\,\delta z\ ,\quad \delta z(0) = \delta z_0$$

$$\Rightarrow\ \delta z(t_f) = \frac{\partial z}{\partial z_0}(t_f, x_0, p_0).\delta z_0 = \exp(t_f A)\,\delta z_0 = \begin{pmatrix} e^{-t_f} & \mathrm{sh}(t_f)\\ 0 & e^{t_f}\end{pmatrix}\delta z_0\ ,$$

so $S'(p_0) = \pi_x\left(\frac{\partial z}{\partial p_0}\right) = \pi_x\left(\frac{\partial z}{\partial z_0}.(0,1)\right) = \pi_x\left(\mathrm{sh}(t_f), e^{t_f}\right) = \mathrm{sh}(t_f)$!

A word on the Lagrange multiplier λ

In the general case, we have $S(p_0,\lambda) = \begin{pmatrix} c\circ\pi_x(z(t_f, x_0, p_0)) \\ \pi_p(z(\dots)) - J_c(\dots)^T\lambda\end{pmatrix}$.

but usually λ may be eliminated by hand.

For instance, consider $c(x) = \frac{1}{2}\|x\|^2 - 1$, $x\in\mathbb{R}^2$.

We have $p(t_f) = \lambda\, x(t_f)$ and $c(x(t_f)) = 0$ so $\overbrace{S(p_0,\lambda)}^{\in\mathbb{R}^3} = \begin{pmatrix} c(x(t_f))\\ p(t_f) - \lambda x(t_f)\end{pmatrix} \in \mathbb{R}^3$.

However the transversality conditions states that

$$\det(p(t_f), x(t_f)) = p_1(t_f)x_2(t_f) - x_1(t_f)p_2(t_f) = 0$$

and so we can simply solve $S(p_0) = \begin{pmatrix} c(x(t_f))\\ \det(p(t_f), x(t_f))\end{pmatrix} = 0$,

and we get λ by

$$\lambda = \frac{(p(t_f)\,|\,x(t_f))}{\|x(t_f)\|^2} = (p(t_f)\,|\,x(t_f)) \times \frac{1}{2}\ .$$